



A THEOREM FOR LOCAL CONVERGENCE OF SCHRÖDER'S METHOD FOR SIMULTANEOUS FINDING POLYNOMIAL ZEROS OF UNKNOWN MULTIPLICITY

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Abstract: *In this note we present a local convergence theorem for Schröder's iterative method considered as a method for simultaneous finding polynomial zeros of unknown multiplicity. Error estimate is also provided.*

Key words: *Schröder's method, polynomial zeros, multiple zeros, local convergence, error estimates.*

1. Introduction

The problem of solving nonlinear equations and systems of equations is among the most important problems in applied mathematics as well as in many branches of engineering sciences, physics, computer science, astronomy, finance, and so on. The most popular iterative methods in the literature are Newton's method, Halley's method [1] and Chebyshev's method [2]. It is well known that Newton's method converges quadratically while Halley's and Chebyshev's methods converge cubically to a simple zero, but all of these methods converge only linear to a multiple zero. It is known that, if the multiplicity m of the zero is known, then the convergence rate of an iterative method can be restored by applying this method to f^{m-1} or $f^{1/m}$ instead of f . In 1870, Schröder [3] has presented the following iterative method for solving nonlinear equations

$$x_{k+1} = x_k - \frac{f(x_k)f'(x_k)}{f'(x_k)^2 - f(x_k)f''(x_k)}, \quad k = 0, 1, 2, \dots$$

which is quadratically convergent and does not require the knowledge of the multiplicity of the zeros.

On the other hand, there are a lot of scientific works devoted to numerical methods for extracting the zeros of polynomials (see, e.g., Pan [4], McNamee [5] and references therein). Note that the best convergence result for Schröder's method for polynomial zeros of unknown multiplicity is due to Proinov and Ivanov [6].

Throughout this work $(K, |\cdot|)$ denotes an arbitrary normed field and $K[z]$ denotes the ring of polynomials over K .

In 2002, Batra [7] has established a semilocal convergence theorem for Newton's method considered as a method for finding all simple zeros of a complex polynomial f simultaneously. In 2015, Proinov and Ivanov [8] have proved local and semilocal convergence theorems for Halley's method as a method for simultaneous computation of all simple zeros of a polynomial f over an arbitrary normed field K . Recently, Kyncheva, Yotov and Ivanov [9] have proved local convergence theorems for Newton, Halley and Chebyshev methods considered as methods for simultaneous determination of all multiple zeros of a polynomial f over an arbitrary normed field K .

In this note, motivated by the above mentioned works, we establish a local convergence theorem with error estimate for Schröder's method considered as a method for simultaneous computation of polynomial zeros of unknown multiplicity.

2. Main result

Let $f \in K[z]$ be a polynomial of degree $n \geq 2$. We consider the zeros of f as a vector in K^n . Namely, a vector $\xi \in K^n$ is called a root-vector of f if f can be presented in the form

$$f(z) = a_0 \prod_{j=1}^n (z - \xi_j),$$

where $a_0 \in \mathbb{K}$.

Furthermore, the vector space \mathbb{K}^n is equipped with the maximum norm defined by

$$\|x\| = \max\{|x_1|, \dots, |x_n|\}.$$

Besides, we define the function $\delta: \mathbb{K}^n \rightarrow \mathbb{R}_+$ by

$$\delta(x) = \min_{i \neq j} |x_i - x_j|.$$

Define the Schröder's iteration in \mathbb{K}^n by

$$x^{k+1} = S_f(x^k), \quad (0.49)$$

where the Schröder's operator S_f is defined by

$$S_f(x) = (S_1(x), \dots, S_n(x)) \text{ with}$$

$$S_i(x) = x_i - \left(\frac{f'(x_i)}{f(x_i)} - \frac{f''(x_i)}{f'(x_i)} \right)^{-1}.$$

Theorem 1. Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ which splits over \mathbb{K} and let $\xi \in \mathbb{K}^n$ be a root-vector of f . Suppose a vector $x^0 \in \mathbb{K}^n$ satisfies the following initial condition

$$\|x^0 - \xi\| < \frac{2\delta(\xi)}{n+1 + \sqrt{n^2 + 6n - 7}}.$$

Then the Schröder's iterative sequence (0.49) is well defined and converges quadratically to ξ with the following error estimate

$$\|x^k - \xi\| \leq \lambda^{2^k - 1} \|x^0 - \xi\| \text{ for all } k \geq 0,$$

where $\lambda = \phi\left(\frac{\|x^0 - \xi\|}{\delta(\xi)}\right)$ and ϕ is a real function

defined by

$$\phi(t) = \frac{(n-1)t}{(2-n)t^2 - 2t + 1}.$$

The main result of this note (Theorem 1) will be proved elsewhere.

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